FUNCTIONS AS RELATIONS ON SETS

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Although you're probably familiar with the concept of a function in mathematics, a function can be defined precisely in terms of a relation between two sets. In technical terms, a *function* is a relation $b = f(a)$ from a set A to another (possibly the same) set B that satisfies two conditions:

(1) The relation must exist for every element $a \in A$. This is the completeness condition.
(2) If two elements $a, b \in A$ are equal, then $f(a) = f(b)$. That is, the equal elements in A cannot map to more than one element in B. This is the uniqueness condition.

Example 1. Let

$$\begin{aligned} A &= \{a, b, c, d\} \\ B &= \{3, 4, 5, 6, 7\} \end{aligned} \tag{1}$$

The relation

$$\mathcal{R}_1 = \{(a,3),(b,5),(c,7),(d,3)\} \tag{2}$$

is a function because every element of A is included in $\mathcal{R}_1$ and there are no repeated elements as the first element in the relation. Note that it is acceptable that two or more distinct elements of A can map to the same element in B. In this example, both a and d map to 3. It is also not necessary for every element in the destination set B to be used.

The set A is called the *domain* of the function f. The element $b = f(a)$ is the *image* of a under the function b. The set of all images of f is called the *range* of f. Note that the range need not include all the elements of B, but in general the range is a subset of B.

Example 2. Using the same two sets 1 as in Example 1, the relation

$$\mathcal{R}_2 = \{(a,3),(a,4),(b,5),(c,7),(d,3)\} \tag{3}$$

is not a function since the image of a is not unique (it maps to both 3 and 4).

Example 3. Again with the sets 1, the relation

$$\mathcal{R}_3 = \{(a,3),(b,4),(d,7)\} \tag{4}$$

is not a function, since it doesn't map all the elements in A (c is missing).

Example 4. Define the set of rational numbers $\mathbb{Q}$ as

$$\mathbb{Q} \equiv \left\{\frac{m}{n} : m,n \in \mathbb{Z} \wedge n \neq 0\right\} \tag{5}$$

Then let

$$f\left(\frac{a}{b},\frac{c}{d}\right) = \frac{ad+cb}{bd} \tag{6}$$

This is a function from the set $\mathbb{Q}^2$ to $\mathbb{Q}$. To verify this, we note that it is defined for all pairs of rational numbers, so it is complete. To prove uniqueness, suppose we have numbers a', b', c' and d' such that

$$\begin{aligned} \frac{a'}{b'} &= \frac{a}{b} \\ \frac{c'}{d'} &= \frac{c}{d} \end{aligned} \tag{7}$$

For example, we might have $\frac{2}{4} = \frac{1}{2}$. Then we have

$$f\left(\frac{a'}{b'},\frac{c'}{d'}\right) = \frac{a'd'+c'b'}{b'd'} \tag{8}$$

$$= \frac{a'd'}{b'd'} + \frac{c'b'}{b'd'} \tag{9}$$

$$= \frac{a'}{b'} + \frac{c'}{d'} \tag{10}$$

$$= \frac{a}{b} + \frac{c}{d} \tag{11}$$

$$= \frac{ad+cb}{bd} \tag{12}$$

Thus a given pair of rational numbers, when reduced to lowest terms, always maps to the same value under f, so f is a function.

The *composition* of two functions f and g, written as $g \circ f$ or $g(f(a))$ is defined as follows. Suppose that f has as its domain the set A and range the set B. Further, suppose that B is a subset of the domain C of g and that g has range D. Then $g \circ f$ is a map from the domain A of f to the range D of g. We can verify that $g \circ f$ is indeed a function by showing it satisfies the two conditions given above. First, since A is the domain of f, it is also the domain of $g \circ f$, so the completeness condition is satisfied.

To show uniqueness, since f is a function, any elements $a_1, a_2 \in A$ with $a_1 = a_2$ are mapped to exactly one element in the domain B of f. Also, because g is a function, any elements $b_1, b_2 \in B$ with $b_1 = b_2$ are mapped to exactly one element in D. Thus equal elements in A will always be mapped to the same element in D under $g \circ f$, so $g \circ f$ is a function.

Example 5. Suppose we have the sets

$$\begin{aligned} A &= \{a, b, c, d\} \\ B &= \{3, 5, 7\} \\ C &= \{1, 3, 5, 7\} \\ D &= \{j, k, \ell, m\} \end{aligned} \tag{13}$$

We then define

$$f = \{(a,3), (b,5), (c,7), (d,3)\} \tag{14}$$

$$g = \{(1,j), (3,k), (5,\ell), (7,m)\} \tag{15}$$

Then

$$g \circ f = \{(a,k), (b,\ell), (c,m), (d,k)\} \tag{16}$$

This follows because, for example, f maps a to 3 and g maps 3 to k, and so on. Here, A is the domain of f and hence also of $g \circ f$. The range B of f is a subset of the domain C of g, and D is the range of g. Although the domain C of g must be covered entirely in the definition of g itself, only a subset of C is needed for the domain of $g \circ f$.

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